Groupoid categories with families

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The goal

There are different categorical notions of model of type theory:

- categories with families
- comprehension categories
- display map categories
- etc.

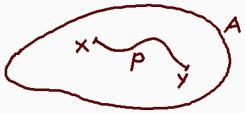
Goal: Compare these notions of model of type theory in univalent foundations

Homotopy type theory (HoTT)

Foundation of mathematics

Internal language of spaces or ∞-groupoids

A type is a space, a term is a point in the space, an equality between two terms is a path



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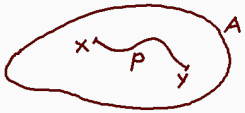

H-levels:

- proposition: every element is equal
- sets: equalities are propositions
- 1-types (synthetic groupoids): equalities are sets
- etc.

Why HoTT?

Functions are automatically functorial, natural, etc.

Univalence: Equivalent types are equal

Structure identity principle: Equivalent structures are equal

Everything is invariant under equivalence

Groupoid categories with families

A category with families (CwF) consists of:

- a category $\mathcal{C}$
- a presheaf $Ty : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$
- a presheaf $Tm : \int Ty^{\mathrm{op}} \to \mathrm{Set}$
- and more

[1] Altenkirch, Kaposi, and Xie (2026). The Groupoid-Syntax of Type Theory Is a Set.

[2] Damato (2026). Containers, Cubes, and Coherences: Containers in Homotopy Type Theory.

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A groupoid CwF consists of:

- a category $\mathcal{C}$
- a pseudofunctor $Ty : \mathcal{C}^{\mathrm{op}} \to 1\text{-Type}$
- a presheaf $Tm : \int_{\mathcal{C}^{\mathrm{op}}} Ty \to \mathrm{Set}$
- and more

Examples:

- sets[1]
- containers[2]

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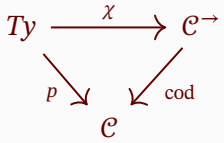
Comprehension categories



Comparison:

$$
\begin{array}{ccc}
\mathrm{GpdCwF} & \simeq & \mathrm{CompCat}_{\mathrm{cons}} \\
\downarrow & & \downarrow \\
\mathrm{CatCwF} & \simeq & \mathrm{CompCat} \\
\uparrow & & \uparrow \\
\mathrm{GpdCwF}_{\text{term-univ.}} & \simeq & \mathrm{CompCat}_{\mathrm{full}}
\end{array}
$$