Higher coinductive types

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Higher inductive types

Inductive type with path constructors:

$$\begin{aligned} S^1 &: \mathcal{U} \\ \mathrm{pt} &: S^1 \\ \mathrm{loop} &: \mathrm{pt} =_{S^1} \mathrm{pt} \end{aligned}$$

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Can we dualize the notion to get higher coinductive types?

Higher coinductive types?

Coinductive type with path destructors?

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Coinductive type with path destructors?

$$
\begin{aligned}
T &: \mathcal{U} \\
\mathrm{node} &: T \to A \\
\mathrm{left} &: T \to T \\
\mathrm{right} &: T \to T \\
\mathrm{neq} &: \prod (t : T), \mathrm{left}(t) =_T \mathrm{right}(t) \to \mathbb{0}
\end{aligned}
$$

Higher coinductive types?

Coinductive type with path destructors?

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\end{aligned}$$

It's inconsistent: We can produce an element of $\mathbb{0}$.

Higher coinductive types? (attempt 2)

Path destructors with unconstrained endpoints?

$$\begin{aligned} T &: \mathcal{U} \\ \mathrm{des} &: \prod(t, u : T), t =_T u \to A \end{aligned}$$

Higher coinductive types? (attempt 2)

Path destructors with unconstrained endpoints?

$$
\begin{aligned}
T &: \mathcal{U} \\
\mathrm{des} &: \prod (t, u : T), t =_T u \to A
\end{aligned}
$$

It's trivial: The type of des is equivalent to $T \to A$.

Can we have higher coinductive types?

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No :(

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No :(

But what if we switch out the identity type?

Parametric observational type theory (POTT)

~~Identity type~~ Bridge type:

$$\mathrm{refl} : \prod(a : A), a \frown_A a$$

$$\begin{aligned}
(a, b) \frown_{A \times B} (a', b') &\text{“=”} (a \frown_A a') \times (b \frown_B b') \\
f \frown_{A \to B} f' &\text{“=”} \left(\prod(a, a' : A), a \frown_A a' \to f(a) \frown_B f'(a') \right) \\
A \frown_{\mathcal{U}} B &\text{“=”} (A \times B \to \mathcal{U})
\end{aligned}$$

Internal parametricity: We can prove that the type $\prod(A : \mathcal{U}), A \to A$ has exactly one element.

Higher coinductive types (in POTT)

Coinductive types with bridge destructors:

$$\begin{aligned} T &: \mathcal{U} \\ \mathrm{des} &: \prod(t, u : T), t \frown_T u \to A \end{aligned}$$

Use case for higher coinductive types

Fibrancy structure:

$$\begin{aligned}
\text{is-fib} &: \mathcal{U} \to \mathcal{U} \\
\text{transp} &: \prod (p : A \frown_{\mathcal{U}} B), f \frown_{p}^{\text{is-fib}} g \to A \to B \\
\text{fill} &: \prod (p : A \frown_{\mathcal{U}} B)\,(q : f \frown_{p}^{\text{is-fib}} g)\,(a : A), a \frown_{p} \text{transp}(p, q, a) \\
\text{id} &: \prod (p : A \frown_{\mathcal{U}} B), f \frown_{p}^{\text{is-fib}} g \to \prod (a : A)\,(b : B), \text{is-fib}(a \frown_{p} b)
\end{aligned}$$

Goal

Semantics for higher coinductive types

Fibrancy closed under type formers, crucially $\text{is-fib}\Big(\sum(A : \mathcal{U}), \text{is-fib}(A)\Big)$

Higher observational type theory